Small signal theory of an E×B drifting electron laser

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The concept of the drifting electron laser (DEL), powered by a relativistic beam of **E**×**B** drifting electrons in crossed electric and magnetic fields, is introduced. The wiggling motion is generated by adding a periodic modulation in either **E** or **B**. In contrast to free electron lasers (FELs) converting kinetic energy and momentum into radiation, the emitted radiation energy and momentum in a DEL come respectively from the change in the electrostatic energy $eE_0\delta X$ and vector potential $eB_0\delta X$ of the electron, δX being the quantum recoil of the guiding center (GC) location perpendicular to the drift direction. The difference between stimulated emission and absorption responsible for the gain is provided by the transverse gradient of the wiggler strength, and the gain curve is *symmetric* relative to the frequency detuning $\delta\omega$. Since the drift velocity and the resonance condition are energy independent, one avoids the low efficiency limits placed on FELs from energy detuning and thermal spreads. Beam energy spreads turn into spreads in the GC location, reducing the gain sensitivity to the beam quality. Saturation in a DEL occurs via the off-axis walk of the emitting electrons. Overlap between the beam and the radiation is maintained by a small tilt of the resonator axis relative to the **E**×**B** drift direction. [S1063-651X(97)09502-0]

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I. INTRODUCTION

In a free electron laser [1–4] (FEL) a relativistic electron beam passing through a periodic magnetic field (wiggler) of wavelength λ_w emits radiation with wave number k_r that is doubly relativistically upshifted relative to the wiggler, $k_r=2\gamma_z^2k_w$, where $k_w=2\pi/\lambda_w$. The emitted radiation energy and momentum come from the beam kinetic energy and momentum. Since the wave-particle resonance condition depends on the beam energy, the detuning resulting from the energy exchange places a limit on the electronic efficiency $\eta\simeq\Delta\gamma/\gamma_0=\frac{1}{2}N_w$. Given that the number of wiggler periods N_w must be large on grounds of per-pass gain, the inherent FEL efficiency is limited to a few percent [1–3]. The sensitivity of the wave-particle resonance on energy also places stringent limits on the FEL tolerance to beam thermal spreads $\langle\gamma-\gamma_0\rangle/\gamma_0\ll\Delta\gamma/\gamma_0=\frac{1}{2}N_w$. Though the efficiency can be improved by tapering the wiggler parameters, the sensitivity to energy spreads cannot.

In the drifting electron laser (DEL) introduced in the present paper, stimulated emission is produced by passing through a wiggler a beam of electrons undergoing a relativistic **E**×**B** drift in orthogonal static electric and magnetic fields. The undulation is provided by adding a periodic modulation in either **B** or **E** (Fig. 1). Though the relativistic frequency upshifting is the same as in a FEL, the drifting electron laser is different in a number of important aspects. The emitted radiation energy and momentum come, respectively, from the change in the electrostatic energy $eE_0\delta X$ and vector potential $eB_0\delta X$ of the electron, δX being the recoil of the guiding center (GC) location perpendicular to the drift direction. Since the resonance condition depends on the average electron drift velocity $u=cE_0/B_0$, which is independent of the potential energy, no detuning results from the energy loss during the interaction. Also, energy spreads in the injected beam have no direct effect on the wave-particle synchronism; they appear as a spread in the GC location transverse to the direction of propagation. Saturation occurs via the off-axis walk of the emitting electrons. Overlap between the beam and the radiation is maintained by a small tilt of the resonator axis relative to the **E**×**B** drift direction.

This paper studies the small signal DEL gain, ignoring the self-field effects from the rippled electron beam, but including the effect of the unperturbed beam space charge. We adopt the quantum mechanical approach that gives a description of the interaction process on the fundamental level and is better suited to bring out similarities and differences with FELs. A companion paper [5] studies the large signal gain and the saturation efficiency following the classical relativistic description of the resonant Hamiltonian. It is shown there that no inherent gain limitation is placed by the interaction physics itself; the only efficiency limitation is technological and is determined by the maximum potential gradient E_0 that can be sustained across the interaction space. The DEL operation exhibits much higher efficiency and lower sensitivity to beam quality than a FEL.

Schematic illustrations of the DEL concept are shown in Figs. 2(a) and 2(b), respectively, employing modulation in either the magnetic or the electric field,

$$\mathbf{B}=B_0\hat{\mathbf{y}}-B_w[\cosh(k_wx)\cos(k_wz)\hat{\mathbf{x}}+\sinh(k_wx)\sin(k_wz)\hat{\mathbf{z}}],\quad \mathbf{E}=E_0\hat{\mathbf{x}}, \tag{1a}$$

$$\mathbf{B}=B_0\hat{\mathbf{y}},\quad \mathbf{E}=E_0\hat{\mathbf{x}}+E_w[\sinh(k_wx)\sin(k_wz)\hat{\mathbf{x}}+\cosh(k_wx)\cos(k_wz)\hat{\mathbf{z}}], \tag{1b}$$

where the periodic terms are derived from the potentials

$$\mathbf{B}_w=\boldsymbol{\nabla}\times A_w,\quad \mathbf{A}_w=k_w^{-1}B_w\cosh(k_wx)\sin(k_wz)\hat{\mathbf{y}}, \tag{2a}$$

$$\mathbf{E}_w=-\boldsymbol{\nabla}\Phi_w,\quad \Phi_w=-k_w^{-1}E_w\cosh(k_wx)\sin(k_wz). \tag{2b}$$

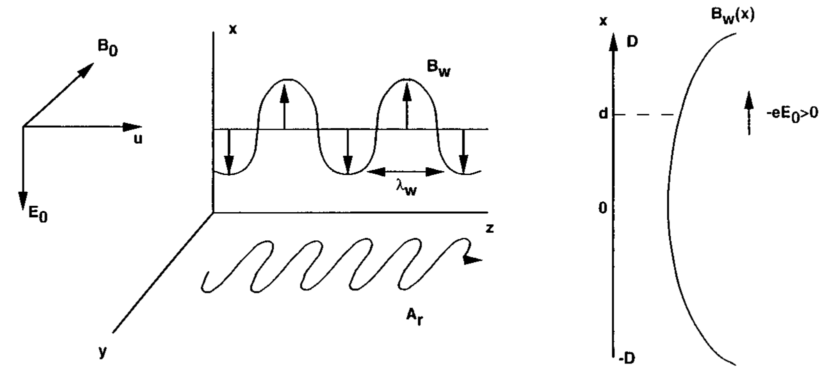

FIG. 1. Schematic illustration of the geometry and field arrangement in a drifting electron laser.

For reasons that will be explained soon the wiggler strength must vary in the x direction, along the uniform static E_0. The interaction Hamiltonians for arrangements (a) and (b), respectively, are

$$H=\sqrt{m^2c^4+c^2\left[\mathbf{P}+\frac{e}{c}\left(\mathbf{A}_0+\mathbf{A}_w+\mathbf{A}_r\right)\right]^2}-e\Phi_0, \quad (3a)$$

$$H=\sqrt{m^2c^4+c^2\left[\mathbf{P}+\frac{e}{c}\left(\mathbf{A}_0+\mathbf{A}_r\right)\right]^2}-e\Phi_0+e\Phi_w, \quad (3b)$$

where $-e$ is the electron charge, $\mathbf{P}=\mathbf{p}-(e/c)\mathbf{A}$ is the canonical momentum,

$$\mathbf{A}_0=B_o x\hat{\mathbf{z}}, \quad \Phi_0(x)=E_o x, \quad (4)$$

and the radiation is a plane wave with vector potential

$$\mathbf{A}_r=A_r\sin(kz-\omega t)\hat{\mathbf{y}}. \quad (5)$$

In the notation of Eq. (4) the dc electric and magnetic fields are negative, $E_0=-E_o$ and $B_0=-B_o$, respectively.

On the quantum description level, the increase in the number of radiation photons is determined by the difference between the stimulated emission and absorption probabilities. In a FEL these probabilities are centered at slightly different frequencies ω_e and ω_a, respectively [4], due to the electron recoil (Compton effect). The difference between stimulated absorption and emission then turns out to be proportional to the derivative of the stimulated emission probability times the difference $\delta\omega\equiv\omega_e-\omega_a$, yielding a gain curve that is *antisymmetric* relative to the frequency shift from resonance [2–4]. In a DEL, the photon emission or absorption process by **E**×**B** drifting electrons is ''recoilless'' and both relevant probabilities peak at the same frequency. The difference between the emission and absorption probabilities, responsible for the gain, is provided mainly by the transverse gradients of the wiggler strength, and, to a lesser degree, from the transverse gradients in the radiation profile and the beam self-field. The gain curve is *symmetric* relative to frequency detuning. The gradient of the wiggler strength, a nuisance in FEL operation, is fundamental for DEL.

The principle of the DEL operation is somewhat similar to magnetron operation where a slow wave $\omega/k\ll c$ is excited by **E**×**B** drifting electrons. The main difference is that in a magnetron a drifting electron can emit a ''slow wave'' caviton without wiggler mediation. In both DEL and magnetrons the basic theory is two-dimensional; the basic FEL description requires only one dimension.

The merits of DEL operation, compared to a FEL of similar operating parameters, are (a) higher small signal gain, for given beam energy, by a factor γ; (b) much higher electronic efficiency that is not limited by the wiggler length (there is no inverse gain-efficiency relation as in a FEL); (c) much smaller sensitivity to thermal beam spreads, independent of wiggler strength or radiation power; (d) prolonging the wave-particle resonance (i.e., ''tapering'') is achieved by merely tilting the radiation beam relative to the drifting beam. A linear radiation focusing mechanism is also introduced by the signal gain.

Though the electrostatic wiggler approach is perhaps easier to implement experimentally, here we will analyze the magnetic wiggler arrangement offering a more obvious similarity with the FEL equations. The quantum approach is used for better exposition of the underlying physics. The rest of the analysis is divided into three parts. Section II introduces the unperturbed eigenfunctions of the relativistic **E**×**B** drifting electrons. Section III computes the stimulated emission or absorption probability. Section IV introduces the effect of the unperturbed beam self-field on the transition probability. Section V combines the previously obtained results into the small signal DEL gain. Section VI discusses the linear self-focusing effect caused by the dependence of the gain on the transverse gradient of the radiation profile. Conclusions follow in Sec. VII.

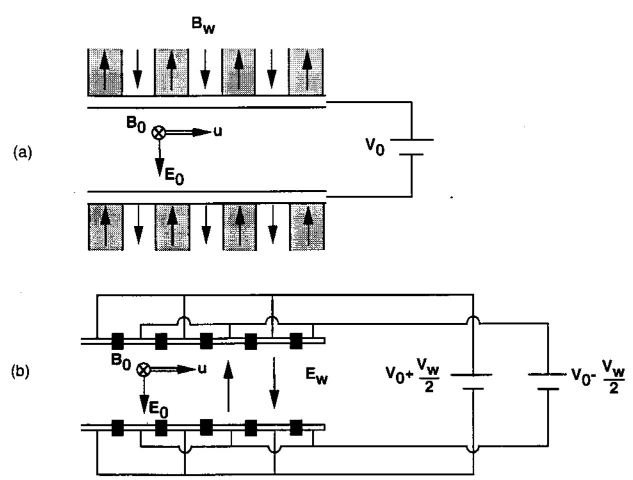

FIG. 2. Illustration of the DEL concept with (a) magnetic and (b) electrostatic wiggler.

II. DRIFTING ELECTRON EIGENSTATES

To discuss radiation emitted from induced transitions between electron states one must first obtain the unperturbed **E**×**B** drifting electron eigenstates. Consider injected electrons that are prepared as eigenstates of the unperturbed Hamiltonian $A_r=0$ involving the only the static fields. Neglecting electron spin, the relativistic Klein-Gordon equation is

$$\left(i\hbar\frac{\partial}{\partial t}+e\Phi_0\right)^2\psi=\left\{m^2c^4-c^2\hbar^2\frac{\partial^2}{\partial x^2}+\tfrac{1}{2}e^2Aw^2\cosh^2(k_wx)+c^2\left(-i\hbar\frac{\partial}{\partial z}-\Omega x\right)^2\right\}\psi, \quad (6)$$

where we have used $\mathbf{P}=-i\hbar\boldsymbol{\nabla}$ and $\mathcal{H}=i\hbar(\partial/\partial t)$ for the canonical momentum and energy operators, respectively. Because of the y invariance, the y momentum can always be set to zero by proper shift of origin, which has resulted in $\mathcal{P}_y\psi=-i\hbar\partial\psi/\partial y=0$ inside Eq. (6). In addition, only the z-averaged rms wiggler strength $\overline{A_w^2}(x)=(1/2)A_w^2\cosh^2(k_wx)$ is kept in the unperturbed motion, allowing invariance along the drift direction z. Hence ψ is an eigenfunction of the energy and z-momentum operators, $\mathcal{P}_z\psi=\hbar q_z\psi$, $i\hbar(\partial\psi/\partial t)=\mathcal{E}\psi$,

$$\psi(x,y,z,t)=e^{-i\mathcal{E}t/\hbar}e^{iqz}\phi(x), \quad (7)$$

where from now on q stands for q_z. It then follows that

$$(\mathcal{E}+eE_0x)^2\phi=\left(m^2c^4-c^2\hbar^2\frac{\partial^2}{\partial x^2}+m^2c^2\Omega^2x^2+c^2\hbar^2q^2-2mc^2\hbar q\Omega x+e^2\overline{A_w^2}\right)\phi. \quad (8)$$

Defining the drift velocity $u=cE_0/B_0$, the relativistic factor $\gamma_u=\sqrt{1-(u/c)^2}$ and the cyclotron frequency in the drifting frame $\hat{\Omega}\equiv\Omega/\gamma_u$ with $\Omega=eB_o/mc$, Eq. (8) becomes after term rearrangements

$$(\mathcal{E}^2-c^2\hbar^2q^2+m^2c^2\hat{\Omega}^2X_q^2-e^2\overline{A_w^2}X_q)\phi=\left[m^2c^4+c^2\left(-\hbar^2\frac{\partial^2}{\partial r^2}+m^2\hat{\Omega}^2r^2\right)\right]\phi. \quad (9)$$

Above, we have set $A_w(x)\simeq A_w(X_q)$ due to the small size of the wave function compared to the wiggler scale 1/kw. The right-hand side (rhs) is the operator for the relativistic harmonic oscillator with energy spectrum

$$\mathcal{E}_n^2=m^2c^4+2(n+\tfrac{1}{2})\hbar\hat{\Omega}, \quad (10)$$

where the wave function ψ is expressed in terms of the distance $r=x-X_q$ from the GC location X_q defined by

$$X_q=\frac{\mathcal{E}eE_o-mc^2\Omega\hbar q}{m^2c^2\hat{\Omega}^2}. \quad (11)$$

Thus Eq. (9) is the quantum description of the cyclotron rotation about a GC drifting along z. The classical Larmor radius ρ_n is related to the rms size of the wave function in the direction transverse to the **E**×**B** motion and is given by $(n+1/2)\hbar\hat{\Omega}=(1/2)m\hat{\Omega}^2\langle\rho_n^2\rangle$. The wave function $\psi_{n,q}(x,z;t)$ for a drifting electron eigenstate with quantum numbers n,q, is

$$\psi_{n,q}(x,z;t)=\phi_n(x-X_q)e^{iqz}e^{-i\mathcal{E}t/\hbar}, \quad (12)$$

where ϕ_n is given in terms of Hermite functions, $\phi_n(r)=\alpha^{-1/4}\exp(-r^2/\alpha)H_n(r/\sqrt{\alpha})$, $\alpha=\sqrt{\hbar/2m\hat{\Omega}}$. Substituting Eq. (12) inside Eq. (9) and solving for the energy $\mathcal{E}$ yields

$$\mathcal{E}_{n,q}=\hbar qu+\frac{1}{\gamma_u}\sqrt{m^2c^4+2mc^2(n+\tfrac{1}{2})\hbar\hat{\Omega}+e^2\overline{A_w^2}}. \quad (13)$$

Expressing q in terms of X_q from Eq. (11) gives the equivalent energy definition

$$\mathcal{E}_{n,q}=\gamma_u\gamma_\perp mc^2-eE_oX_q, \quad (14)$$

$$\gamma_\perp=\sqrt{1+2(n+1/2)\hbar\hat{\Omega}/mc^2+(e\overline{A_w}/mc^2)^2}. \quad (15)$$

The last term in the rhs of Eq. (14) is the potential energy at the GC location. In the nonrelativistic limit the $\gamma_u\gamma_\perp$ term breaks into the sum of the GC drift kinetic energy $(1/2)mu^2$ plus the cyclotron rotation energy $(n+\frac{1}{2})\hbar\hat{\Omega}$ about the GC. Substituting Eq. (14) in the definition of the GC (11) yields the conserved momentum $P_z=\hbar q$ as the sum of the kinematic and the vector potential momentum of the GC location,

$$P_z=\hbar q=\gamma_u\gamma_\perp mu-m\Omega X_q. \quad (16)$$

Eventually the energy and momentum have been expressed in terms of the vector potential and the ES potential at the center of the wave function (the classical GC), plus the cyclotron rotation energy. Note that the z-momentum quantum number q uniquely defines the GC location X_q and vice versa, henceforth the subscript (q) is dropped from X for simplicity.

III. INTERACTION WITH RADIATION

We now describe the interaction of the unbound electron eigenstates with the cavity radiation field. Adopting the old quantum treatment the cavity modes are given by the classic solutions of Maxwell equations. The first order perturbed Hamiltonian is

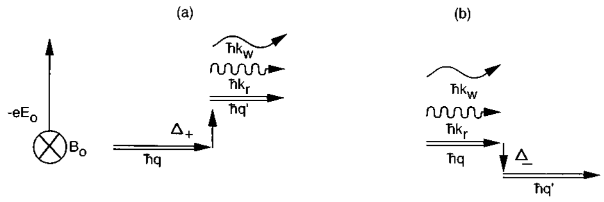

FIG. 3. Diagram for the elementary process of photon emission or absorption. Instead of velocity recoil, a ''parallel shifting'' of the electron orbit in space by $\delta X=\hbar\omega_r/eE_o$ occurs.

$$\mathcal{H}_1=\frac{1}{4}\frac{mc^2}{\gamma_u\gamma_\perp}\left\{\left(\frac{eA_w}{mc^2}\right)^2\cosh^2(k_wx)\times[e^{2ik_wz}+\text{c.c.}]+\left(\frac{eA_w}{mc^2}\right)\left(\frac{eA_r}{mc^2}\right)\cosh(k_wx)\times[e^{i(k_r+k_w)z-i\omega_rt}+e^{i(k_r-k_w)z-i\omega_rt}+\text{c.c.}]\right\}. \quad (17)$$

The first term in the right-hand side, coming from the difference $A_w^2(x,z)-A_w^2(x)$, is time independent and yields successive scatterings of the drifting electrons off the wiggler periodic potential; it is the quantum analog to the classical nonlinear wiggling motion. The interaction of the fundamental wiggling motion with the radiation is given by the second term; the plus sign k_r+k_w corresponds to upshifted radiation frequency, the case of interest. The interaction Hamiltonian for the stimulated emission is thus taken as

$$\mathcal{H}_1=\frac{mc^2}{4\gamma_u\gamma_\perp}a_wa_r\cosh(k_wx)[e^{i(k_r+k_w)z-i\omega_rt}+\text{c.c.}], \quad (18)$$

where, as usual, $a_w=eA_w/mc^2$, $a_r=eA_r/mc^2$.

The process of a photon emission or absorption is depicted in Fig. 3. The energy-momentum conservation yields [6]

$$\mp\hbar\omega_r=\hbar u(q'-q)\pm(n'-n)\hbar\hat{\Omega}/\gamma_u, \quad (19a)$$

$$\mp\hbar(k_r+k_w)=\hbar(q'-q). \quad (19b)$$

The periodic wiggler introduces the factor $\hbar k_w$ in the momentum balance, since in a periodic medium momentum is conserved within $\hbar k_w$. It is the wiggler mediation that makes the photon emission possible; recall that an electron free from external fields can only scatter a photon. The emitted frequency, in general, depends on whether transitions between cyclotron states take place. We focus on transitions between states of equal cyclotron energy $n'=n$,

$$\mathcal{E}_{n,q'}-\mathcal{E}_{n,q}=(q'-q)\hbar u=\mp\hbar\omega_r. \quad (20)$$

From Eqs. (19b) and (20) it follows that the emitted frequencies are centered at

$$\omega_r-u(k_w+k_r)=0. \quad (21a)$$

Thus

$$\omega_r=\left(1+\frac{u}{c}\right)\gamma_u^2k_wu. \quad (21b)$$

For $u\simeq c$ Eq. (21b) yields $\omega_r=2\gamma_u^2k_wc$, corresponding to the usual FEL operation range. Conservation of the total momentum, using the relation (16) between $\hbar q$ and the GC location, requires that the electron GC recoil by

$$\delta X=\frac{\hbar(q'-q)}{m\Omega}=\mp\frac{\hbar(k_w+k_r)}{m\Omega}. \quad (22)$$

The direction of this recoil is perpendicular to the direction of the drift **u** and across the magnetic field. Substituting $\hbar(q'-q)=\mp\hbar\omega_r/u$ from the energy conservation equation (20) into Eq. (22) yields

$$\mp\hbar\omega_r=-m\Omega u\,\delta X=-eE_o\delta X=\delta(-e\Phi_0). \quad (23)$$

The exchanged radiation energy equals the change in the electrostatic energy [6] of the electron GC. In a similar manner the change in the radiation momentum equals the change in the canonical momentum

$$\mp\hbar k_r=\delta P_z=-m\Omega\,\delta X=\delta(eA_z/c), \quad (24)$$

stemming from the GC displacement across the vector potential. Stimulated emission in crossed E and B fields involves changes in the electrostatic and vector potential only. The kinematic energy and momentum remain invariant during the transition.

Whether emitted radiation is amplified depends on the relative strength between absorption and emission probabilities. According to Eqs. (23) and (24) the center X of the wave function $\phi_n(x-X)$ is shifted by $\delta X=\pm\Delta$ with

$$\Delta=\frac{\hbar\omega_r}{eE_o}=\frac{\hbar\omega_r}{m\Omega u} \quad (25)$$

during emission or absorption. The per unit time change in the probability amplitude $p_\pm(t)$ for emission or absorption is written in terms of $r=x-X$ and $r'=x-(X\pm\Delta)=r\mp\Delta$ as

$$p_\pm=\left\langle\phi_n(r\mp\Delta)\left|\frac{i}{\hbar}V_1\cosh[k(X+r)]\right|\phi_n(r)\right\rangle\times\delta_{\mathbf{q}'-\mathbf{q}\mp(\mathbf{k}_r+\mathbf{k}_w)}e^{i(\mathcal{E}_{n,q'}-\mathcal{E}_{n,q})t/\hbar\mp i\omega_rt}, \quad (26)$$

where $V_1=a_wa_rmc^2/4\gamma_\perp\gamma_u$. The right-hand side involves the overlap integral between initial and final states, shown in Fig. 4 for the ground cyclotron state $n=0$. Expression (13) is computed by expanding $\phi_n(r\mp\Delta)=\phi_n(r)\mp(d\phi_n/dr)\Delta+\cdots$ and $V_1\cosh(k_wx)=V_1\cosh(k_wX)+k_wV_1\sinh(k_wX)r+\cdots$, and expressing r, d/dr as

$$r=\sqrt{\frac{\hbar}{2m\Omega}}(a+a^\dagger),\quad \frac{d}{dr}=\frac{i}{\hbar}\mathcal{P}_r=\sqrt{\frac{m\Omega}{2\hbar}}(a-a^\dagger), \quad (27)$$

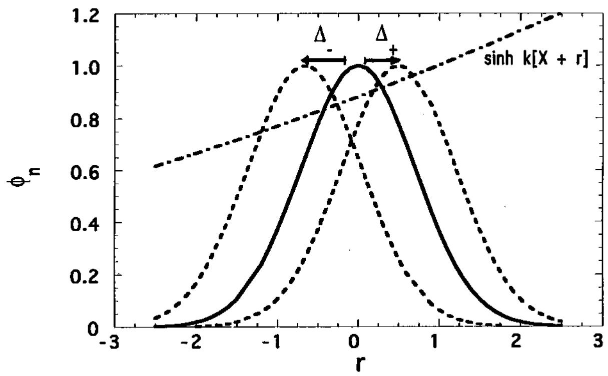

FIG. 4. Wave function $\phi_0(r)$ before (solid) and after (dashed) transition. r is in units of $\sqrt{\hbar/2m\Omega}$. The broken line is the wiggler strength profile in the vicinity of X (not to scale).

$a,a^\dagger$ being the Heisenberg operators with the properties $a\,\phi_n=\sqrt{n}\,\phi_{n-1}$, $a^\dagger\,\phi_n=\sqrt{n+1}\,\phi_{n+1}$. The total probability for electron transition in the lower (upper) energy state after time t,

$$W_\pm(t)\equiv\left|\int_0^t dt'\,p_\pm(t')\right|^2,$$

is found from Eqs. (26) and (27) and the orthonormality among ϕ_n,

$$W_\pm=\frac{1}{\hbar^2}\Bigg\{A^2+AA''\langle\phi_n|r^2\phi_n\rangle\mp(A^2)'\Delta\langle\phi_n'|r\phi_n\rangle+\frac{\Delta^2}{2}[(A')^2\langle\phi_n'|r\phi_n\rangle^2+2A^2\langle\phi_n''|\phi_n\rangle+AA''\langle\phi_n''|\phi_n\rangle\times\langle\phi_n|r^2\phi_n\rangle]\Bigg\}\frac{\sin^2[\{\omega_r-(k_r+k_w)u\}t/2]}{[\{\omega_r-(k_r+k_w)u\}/2]^2},\qquad(28)$$

where $A\equiv V_1\cosh(k_wX)$, $(')\equiv d/dX$ and we have omitted terms higher than Δ^2, r^2. The net emission probability

$$W=W_+-W_-=-\frac{1}{\hbar^2}2\Delta(A^2)'\langle\phi_n'|r\phi_n\rangle+O(\Delta^3)\qquad(29)$$

is proportional to the GC displacement Δ times the transverse gradient of the wiggler strength.

IV. EFFECT OF THE BEAM FIELD

We have so far ignored the influence of the electrostatic field from the charged electron layer. To take self-fields into account consider a monolayer of **E**×**B** drifting electrons with GC located at X. Let the charge density $\rho(r)=-\sigma|\Psi(r)|^2$ be a Gaussian of total surface charge density $-\sigma$. The system wave function Ψ is then a superposition of eigenstates of various n's,

$$\Psi(r)=\left(\frac{1}{\sqrt{\pi}d}\right)^{1/2}e^{-r^2/2a^2}=\sum_n c_n\phi_n(r),\qquad(30)$$

subject to $\Sigma_n|c_n|^2=1$. We take the width d of the Gaussian much less than the "ground state Larmor radius," $d=\zeta(\hbar/m|\Omega|)^{1/2}$ with $\zeta\ll1$, meaning that the thickness d of the electron layer is smaller than the quantum recoil Δ from Eq. (12). An asymmetry is now introduced in the GC jumps between stimulated absorption and emission due to the difference in the electric strengths above and below the beam,

$$\Delta_\pm=\pm\frac{\hbar\omega_r}{eE_\pm},\qquad(31)$$

where $E_\pm=E_o\pm2\pi\sigma$ and $E_o=\frac{1}{2}(E_++E_-)$ is the field at X. Because $|\Delta_+|<|\Delta_-|$ a stronger overlapping between initial and final states occurs during emission than absorption. Repeating the exercise of the previous paragraph yields an expression similar to Eq. (28) with $\Psi(r)$ in lieu of ϕ_n and $\Delta_\pm$ instead of Δ. Two remarks are in order. First, the beam field modifies the oscillator eigenfunctions and eigenvalues. However the corrections from using perturbed $\tilde{\phi}_n$ and $\tilde{\mathcal{E}}_n$ to construct Ψ are shown to disappear in the classical limit. Second, the no-cyclotron-emission constraint $n'=n$ during transitions leads to $\langle\Psi''|\Psi\rangle=-(m/\hbar^2)\langle\Psi|\mathcal{H}|\Psi\rangle$ instead of the momentum expectation value $-(1/\hbar^2)\langle\Psi|\mathcal{P}^2|\Psi\rangle$. Computing $\langle\Psi|\mathcal{H}|\Psi\rangle=\frac{1}{2}[(\hbar^2/2md^2)+m\Omega^2d^2/2]$ one obtains

$$W_+-W_-=\frac{1}{\hbar^2}\Bigg\{\frac{\Delta_+-\Delta_-}{2}(A^2)'+\frac{\Delta_+^2-\Delta_-^2}{2}\times\Bigg[(A')^2-2A^2\frac{m}{2\hbar^2}\left(\frac{\hbar^2}{2md^2}+\frac{m\Omega^2d^2}{2}\right)-AA''\sum_n|c_n|^2(n+\tfrac{1}{2})^2\Bigg]\Bigg\}\times\frac{\sin^2\{[\omega_r-(k_r+k_w)u]t/2\}}{\{[\omega_r-(k_r+k_w)u]/2\}^2}.\qquad(32)$$

In the low space charge limit, $4\pi|\sigma|/E_o\ll1$, the first contribution from $\Delta_+-\Delta_-\simeq2|\Delta|+O(\sigma^2/E_\sigma^2)$ is similar to that in Eq. (28). In addition, there is a finite contribution from the quadratic GC recoil term,

$$-\frac{\Delta_+^2-\Delta_-^2}{2}\simeq\frac{\hbar^2\omega^2}{e^2E_o^2}\frac{E_+-E_-}{E_o}=4\pi\Delta^2\frac{\sigma}{E_o}.\qquad(33)$$

An interesting twist in the theory is that when the beam width is much larger than the recoil size, $d\gg\Delta$, the space charge correction in the electron recoil distance inside the beam is of higher order in $\hbar$ and the gain contribution from $\Delta_+^2-\Delta_-^2$ vanishes in the classical limit. Consider, for example, a uniform GC distribution of density n_0 with $d\gg\Delta$ (here the finite Larmor radius is irrelevant as long as it is smaller than d). Energy conservation during the transition (23) in the presence of the local beam self-field $-eV_b=eE_oX-2\pi e^2n_oX^2$ yields

$$\mp\hbar\omega_r=eE_o\Delta-\frac{m}{2}\omega_b^2\Delta^2,\qquad(34)$$

where $\omega_b^2=4\pi e^2n_o/m$ is the beam plasma frequency. The solution yields

$$\Delta_{\pm}=\frac{eE_o/m-\sqrt{(eE_o/m)^2\pm 2\hbar\omega_r\omega_b^2/m}}{\omega_b^2}$$
$$\simeq\pm\frac{\hbar\omega_r}{eE_o}\left[1\mp\frac{\hbar\omega_r}{eE_o}\left(\frac{m\omega_b^2}{eE_o}\right)\right]. \tag{35}$$

The difference $\Delta_+^2-\Delta_-^2$ is of order $\hbar^3$ instead of $\hbar^2$ in Eq. (33) and the corresponding contribution to the gain vanishes in the classical limit. Space charge effects in case of a wide beam enter through the shear in the $\mathbf{E}_0\times\mathbf{B}_0$ velocity. Different beam layers drift at slightly different velocities $u(X)=u_0+(\omega_b^2/\Omega)(X-X_o)$ where X_o is the beam center; the detuning effect from this variation comes into play when the beam thickness exceeds the recoil distance. The detuning from resonance becomes a function of the GC location $X-X_o$. Summing up over the GC distribution in X it turns out (Appendix A) that the space-charge correction to the gain scales as $(\sigma/E_0)^2$ where $\sigma=en_od$.

In conclusion, the space-charge contribution to the gain depends on the field jump across the beam $\delta E_0=4\pi\sigma$ rather than the shear $\delta E_0/d$, provided the beam thickness is much less than the wiggler wavelength λ_ω. For a microscopic beam thickness $d\ll\Delta$ the gain correction is proportional to (σ/E_0); in general, it can be either positive or negative (stabilizing) depending on the sign of E_0. For macroscopic thickness $d\gg\Delta$ the correction scales as $(\sigma/E_0)^2$ and is always negative (stabilizing).

V. PER-PASS GAIN FOR A SHEET BEAM

The radiated power per cavity pass δP, determined by the number of photons emitted after a cavity transit time $t=L/u$, is equal to the electron flux $(\sigma/e)u\alpha$, α being the width of the beam in the y direction, times net emission probability, times the emitted quantum $\hbar\omega_r$,

$$\delta P_r=(\sigma/e)u\alpha[W_+-W_-]\hbar\omega_r. \tag{36}$$

Substituting Eqs. (32) and (33) inside Eq. (36), using d from after Eq. (30), and taking the classical limit $\hbar\rightarrow 0$ two contributions remain in the gain: one independent of the cyclotron energy n, and one proportional to the classical Larmor radius $\langle\rho^4\rangle=(\hbar/m|\Omega|)^2\Sigma_n|c_n|^2(n+\frac{1}{2})^2$. In the ''cold'' beam case $k^2\langle\rho^2\rangle\ll 1$ one obtains, defining the detuning $\Delta\omega\equiv\omega_r-(k_r+k_w)u$,

$$\delta P_r=\frac{k_w\alpha}{8}\frac{\sigma}{em\Omega}\frac{e^4A_w^2A_r^2}{m^2c^4\gamma_u^2\gamma_\perp^2}\left[\sinh(k_wX)\cosh(k_wX)+\frac{\sigma}{E_o}\frac{\omega_r}{k_wu}\frac{\pi}{4\zeta^2}\cosh^2(k_wX)\right]\omega_r^2\frac{\sin^2[\Delta\omega t/2]}{[\Delta\omega/2]^2}. \tag{37}$$

The first and second terms on the right-hand side of Eq. (37) describe two effects that make the probability for stimulated emission larger than stimulated absorption. First, the wiggler strength increases with X and thus favors transitions shifting the GC upwards $\Delta_+>0$. That corresponds to electrons falling into lower potential energy state $\delta(-e\Phi_0)=-eE_o\Delta_+<0$ via radiation emission. Second, the discontinuity of the dc field across the electron beam charge has a similar effect; the resulting difference between stimulated emission and absorption probability is of the order of the GC shift squared Δ^2 and proportional to $1/|E_-|^2-1/|E_+|^2>0$.

Implicit in the derivation of Eq. (37) is the low gain approximation when the wave amplitude does not vary significantly over the interaction length. The total radiation flux in the cavity P is related to the radiation amplitude by $P_r=(c/4\pi)(\omega_r/c)^2A_r^2\pi w_o^2$, where w_o is the waist size for a Gaussian optical beam. Defining the per pass gain $G\equiv\delta P/P$, and expressing the surface charge density in terms of the beam current $\sigma=I_b/\alpha u$ one can cast the per-pass gain in a cavity fed by a sheet current I_b as

$$G=\frac{I_b}{2I_0}\frac{ck_w}{\beta_u\Omega}\frac{a_w^2}{\gamma_u^2\gamma_\perp^2}\frac{1}{(k_rw_o)^2}\left[\frac{1}{2}\left(\frac{1}{a_w^2}\frac{\partial a_w^2}{\partial(k_wX)}\right)+\left(\frac{\delta E_o}{E_o}\right)\frac{\omega_r}{k_wu}\frac{1}{16\zeta^2}\right]\left(\frac{k_rL}{\beta_u}\right)^2\Theta(\xi), \tag{38}$$

where $\beta_u=u/c$, $a_w(X)=(eA_w/mc^2)\cosh(k_wX)$, $I_0=mc^3/e=17.069kA$, the detuning parameter $\xi=\Delta\omega L/2u=(\Delta\omega/\omega_o)(k_rL/2\beta_u)$ and the line shape factor is

$$\Theta(\xi)=\frac{\sin^2\xi}{\xi^2}. \tag{39}$$

The gain profile $\Theta(\xi)$ is *symmetric* relative to the resonant frequency $\omega_0=(k_r+k_w)u$, contrasting the antisymmetric FEL gain that goes as $d\Theta/d\xi$.

Formula (38) emphasizes the gain dependence on the *transverse* gradients da_w/dX and $\delta E_0\propto\sigma$ relative to the wave propagation direction, the emission process being fundamentally two dimensional. Gain results when the strengths a_w and E_0 increase in the direction of the dc electric force. If the space-charge contributions are omitted, the gain is antisymmetric relative to the beam placement from the midplane $X=0$. For the fields shown in Fig. 1 the force is positive $-eE_0=eE_o$ and gain results when the beam is injected above the midplane $X=d>0$; a negative gain of opposite value occurs for $X=-d$. If the direction of E_0 and B_0 is reversed the beam must be injected below the mid plane $X<0$ where $\partial a_w^2/\partial X<0$ is of the same sign with the force $-eE_o$ (the drift velocity u remains the same). In that case the space charge contribution in Eq. (38) changes sign since $\delta E_0=4\pi(-\sigma)$ does not reverse with E_0; a positron beam of opposite charge density is required to yield exactly the same gain under field inversion. This is the well known CP (charge-parity) symmetry of electrodynamics.

It should be pointed out that a microscopic beam thickness, of the order of the ground state Larmor radius $d=\zeta(\hbar/m\Omega)^{1/2}$, cannot be resolved in the classical limit (macroscopic) description. The classical analog in that case is a beam of zero macroscopic thickness $d/\lambda_r=0$, i.e., a δ-function density distribution. The treatment of magnetron mode excitation by a sheet beam [7], a somewhat similar process involving slow wave $\omega/k\ll c$ excitation by $\mathbf{E}\times\mathbf{B}$ drifting electrons, showed that the classical result for a zero-thickness sheet beam agrees with the quantum result with $\zeta=\frac{1}{4}$.

Figure 5 shows the gain G vs detuning parameter ξ, obtained from Eq. (38) over an interaction length L of $N_w=200$

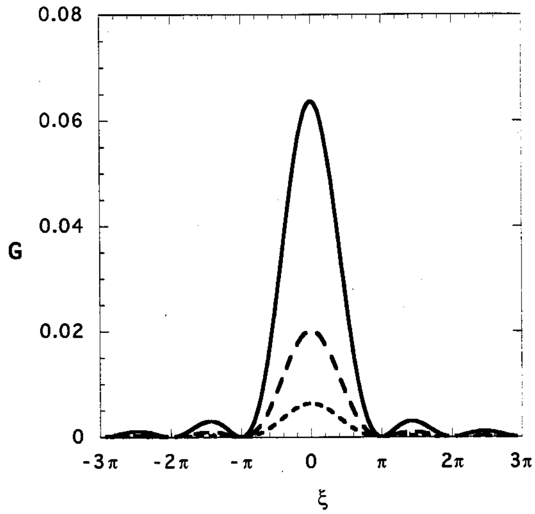

FIG. 5. Plot of gain vs detuning $\xi=(\Delta\omega/\omega_o)(k_r L/2\beta_u)$ for beam currents of 10, 33, and 100 mA over 200 wiggler periods for normalized wiggler strength $a_w=0.5$.

wiggler periods for different beam currents. The beam thickness is taken to be half the ground level Larmor radius, setting $\zeta=\frac{1}{2}$. The other parameters are $a_w=0.10$, $k_w=2\pi\ \mathrm{cm}^{-1}$, $k_w X=1$, $u/c=0.99$, corresponding to a dc electric field $E_0=297$ kV/cm at $B_0=1000$ G. The three gain curves correspond to beam currents I_b equal to 10 mA, 33 mA, and 100 mA, respectively, yielding σ/E_o ratios from 1.02×10^{-6} to 1.02×10^{-5}. The natural half-width of the excited spectrum is inversely proportional to the cavity length

$$\frac{\Delta\omega}{\omega_r}\simeq\frac{\pi}{k_r L}=\frac{1}{\gamma_u^2 N_w}. \tag{40}$$

Over short interaction times $\Delta\omega t\ll1$ where $t=z/u$ the radiation power increases as t^2, and is independent of the detuning $\Delta\omega$.

The computed gain is valid in the low-beam current, *low gain* regime, where coherent beam bunching and excitation of collective beam modes of frequency ω_b can be neglected. Inclusion of the coherent beam bunching in the high gain case leads to exponential growth. Inclusion of collective beam mode excitation in the high-beam current case leads to Raman scattering, during which the stimulated emission becomes a three wave process; viewed in the drifting frame a wiggler ''virtual photon'' of frequency $\omega_w^*=\gamma_u(uk_w)$ stimulates the excitation of a beam quantum $\omega_b^*=\gamma_u(\omega_b-uk_b)$ with the emission of a photon $\omega_r^*=\omega_w^*-\omega_b^*$.

The spectral broadening due to the finite ''lifetimes'' of the interacting radiation–electron eigenstates is obtained by multiplying Eq. (12) with $e^{-\Gamma t/2}$, $1/\Gamma$ being the combined lifetime. The spectral density $\mathcal{U}(\omega)$ of the emitted radiation from the integration $\int_0^\infty dt\ e^{-\Gamma t}[W_+(t)-W_-(t)]$ is

$$\mathcal{U}(\omega)\propto\frac{[\omega_r-(k_r+k_w)u]^2}{[\omega_r-(k_r+k_w)u]^2+\Gamma^2}. \tag{41}$$

For radiation frequencies in the THz range, and for beam densities yielding similar plasma frequencies, the spontaneous emission and e-e collision effects are negligible. The linewidth Γ is determined by the inverse electron transit time t through the cavity length L, and the inverse cavity decay time $\Gamma_Q=\omega/Q$, $\Gamma^{-1}\simeq(\omega/Q)^{-1}+(u/L)^{-1}$. Notice the absence of thermal broadening due to velocity spreads. Since the drift velocity u is determined by the field strengths at the GC location X, any velocity spreads among electrons injected at the same GC location must be distributed in their cyclotron rotation velocities. In quantum terms this translates as a spread over the oscillation quantum number n, Eq. (12). The radiation gain (38) is independent of n and thus of thermal spreads.

VI. INTERACTION BETWEEN THE RADIATION ENVELOPE AND GAIN

So far a uniform radiation amplitude has been assumed during the gain computation. Since the gain was shown to depend on the transverse gradients of the wiggler strength and the electric field, one would like to include similar effects caused by the variation of the radiation amplitude in the transverse direction. Repeating the gain computation with an amplitude profile that varies along x, $A_r\to A_r U(x)$, one simply replaces A in expressions (28) and (29) with $A=V_1\cosh(k_w X)U(X)$, yielding the final result

$$G=\frac{I_b}{2I_0}\frac{ck_w}{\beta_u\Omega}\frac{a_w^2}{\gamma_u^2\gamma_\perp^2}\frac{1}{(k_r w_o)^2}\left[\frac{1}{2}\left(\frac{1}{a_w^2}\frac{\partial a_w^2}{\partial(k_w X)}\right)+\left(\frac{\delta E_0}{E_0}\right)\frac{\omega_r}{k_w u}\frac{1}{16\zeta^2}+\frac{1}{2}\left(\frac{1}{a_r^2}\frac{\partial a_r^2}{\partial(k_w X)}\right)\right]\left(\frac{k_r L}{\beta_u}\right)^2\Theta(\xi), \tag{42}$$

where $a_r(X)=(eA_r/mc^2)U(X)$. An additional term proportional to the transverse gradient in the radiation intensity $\partial a_r^2/\partial X$ has been introduced in the gain. Growth is therefore affected by the beam placement relative to the radiation envelope; placing the electron beam on the positive (negative) slope side of the Gaussian radiation envelope increases (decreases) the local radiation gain.

For a finite thickness electron beam the local gain is higher in the area of increasing and lower in the area of decreasing radiation amplitude, thus a self-focusing mechanism for the radiation results from the direct interaction between gain and intensity gradient. In a DEL self-focusing is a *linear* effect depending on the local *intensity gradient*, while focusing in FELs is nonlinear and depends on intensity. To illustrate the self-focusing effect, consider the following approximation for the wave equation in the presence of the optically active electron beam. In Maxwell's equation $\nabla^2 A_y-(1/c^2)\partial^2 A_y/\partial t^2=-(4\pi/c)j_y$ we let $A_y=A_r U(x,z)e^{ik_r z-i\omega_r t}$ and write the current as $j_y=(j_y E_y^*)/E_y^*$ related to the gain via $j_y E_y^*=(1/4\pi)(\omega_r^2/c^2)A_r^2 U^2 dG/dt$ and $E_y^*=(i\omega/c)A_r U$. The paraxial wave equation then becomes

$$\frac{\partial^2 U}{\partial x^2}+2ik_r\frac{\partial U}{\partial z}=\frac{i}{\omega_r}\frac{\omega_r^2}{c^2}\frac{dG}{dt}U=2i\left(\kappa\frac{\partial U}{\partial x}+gU\right), \tag{43}$$

where

$$\kappa \equiv \frac{I_b}{2I_0}\frac{\omega_r}{\beta_u\Omega}\frac{\omega_r^2}{u^2}z, \quad g \equiv \frac{I_b}{2I_0}\frac{\omega_r}{\beta_u\Omega}\frac{\omega_r^2}{u^2}k_w z. \tag{44}$$

The term proportional to g on the right-hand side gives the usual gain dependence on the beam-radiation overlapping (i.e., ''filling factor'') while the additional focusing term is introduced from the gain dependence on the envelope derivative. Without gain, $\kappa = g = 0$, the eigenmodes of Eq. (43) are the well known Gaussian wave packets for propagation in vacuum,

$$U_0(x,z) = \sqrt{\frac{kb}{\pi}}\frac{i}{z-ib}\exp\left[ik\frac{x^2}{2(z-ib)}\right]. \tag{45}$$

The radiation spot size is given by $w(z) = w_o\sqrt{1+z^2/b^2}$, the radiation waist w_o in the x direction is related to the diffraction parameter b via $w_o^2 = 2b/k$, and we have assumed infinite waist size along y. When the gain is turned on the solution becomes

$$U(x,z) = \exp\left(\frac{g}{k_r}z\right)\exp\left(i\kappa x + i\frac{\kappa^2}{2k_r}z\right)U_0(x,z). \tag{46}$$

The first factor gives the amplitude growth along the interaction space. The next (slowly varying phase) factor introduces a deformation of the spherical wave fronts; the peak amplitude shifts off-axis along the line $x = -(\kappa/2k_r)z$. The above heuristic solution assumed κ=const when in fact κ changes with z. The radiation growth and envelope equations must be solved simultaneously to obtain the exact profile and growth along z.

VII. EFFICIENCY, SENSITIVITY TO BEAM QUALITY AND CONCLUSIONS

We have demonstrated radiation emission from relativistic **E**×**B** drifting electrons at the doubly relativistically upshifted wiggler frequency $k_w u$. Electrostatic energy and magnetic momentum are converted into radiation; the drift velocity remains constant, maintaining the resonance condition along the interaction space. The emission line shape is symmetric relative to the frequency or velocity detuning, a trademark of emission from **E**×**B** drifting electrons. The gain is proportional to the wiggler and electric field gradients orthogonal to the propagation direction. Additional gain dependence on the slope of the radiation envelope introduces a linear self-focusing effect.

The DEL per-pass gain will now be compared to the corresponding FEL gain. Neglecting space-charge and radiation envelop effects from Eq. (42) yields

$$G_{\rm DEL} = \frac{I_b}{2I_0}\frac{ck_w}{\beta_u\Omega}\frac{\hat{a}_w^2}{\gamma_u^2\gamma_\perp^2}\frac{1}{(k_r w_o)^2}\left(\frac{k_r L}{\beta_u}\right)^2\left[\frac{1}{2\hat{a}_w^2}\frac{\partial \hat{a}_w^2}{\partial(k_w X)}\right]\Theta(\xi), \tag{47}$$

where $\hat{a}_w(X) = a_w\cosh(k_w X)$ and $I_0 = mc^3/e = 17.069$ kA. In the same notation

$$G_{\rm FEL} = \frac{I_b}{2I_0}\frac{a_w^2}{\gamma_u^2\gamma_o^3\beta_u^2}\frac{1}{(k_r w_o)^2}\left(\frac{k_r L}{\beta_u}\right)^3\left[-\frac{d}{d\xi}\Theta(\xi)\right], \tag{48}$$

where $\gamma_o^2 = \gamma_u^2 + a_w^2/2$. The gain ratio is

$$\frac{G_{\rm DEL}}{G_{\rm FEL}} \simeq \frac{\gamma_o^3}{k_r L}\frac{ck_w}{\Omega}\tanh(k_w X)\beta_u^2 \geqslant \frac{\gamma_o}{2\pi N_w}, \tag{49}$$

showing that DEL operation favors higher beam energies and shorter wiggler lengths than a FEL.

The fact that the drift velocity is determined by $\mathbf{E}_0\times\mathbf{B}_0$ limits the DEL sensitivity to the thermal beam spreads. In a DEL, the velocity or energy spreads of the injected beam are converted into spreads in the GC location and affect the overlap in real space. The velocity mismatch *per se* is irrelevant, since the GC's of all electrons drift at the same u. Conservation of the canonical energy or momentum between two points of an electron orbit located inside and far outside the wiggler (where $E_0 = B_0 = 0$) shows that an initial mismatch $\delta\gamma \equiv \gamma - \gamma_u$ from the exact resonance causes a GC shift δX from the intended location, and a finite gyroradius, given, respectively, by

$$\delta X \simeq \gamma_u\beta_u\frac{c}{\Omega}\frac{\delta\gamma}{\gamma_u}, \quad \rho = \gamma_u\frac{c}{\Omega}\left(\frac{2\delta\gamma}{\gamma_u}\right)^{1/2}. \tag{50}$$

To maintain good overlapping with radiation one must limit δX, ρ well below the radiation waist w_0 and/or the gap size D. Because $\rho \geqslant \delta X$, the tolerance to thermal spreads is determined by the condition $\rho/D \ll 1$. Letting $\delta\gamma \simeq \langle\gamma - \gamma_u\rangle$ yields

$$\frac{\langle\gamma - \gamma_u\rangle}{\gamma_u} \ll \left(\frac{D\Omega}{c}\right)^2\frac{1}{2\gamma_u^2}. \tag{51}$$

where the right-hand side is given by $1.72\times 10^1 B^2[\mathrm{T}]\ D^2[\mathrm{cm}]/\gamma_u^2$. In a FEL, on the other hand, the sensitivity to beam quality is determined by the energy spread relative to the trapped particle ''bucket'' width $\Delta\gamma = \gamma_u\sqrt{2a_w a_r}$; the tolerated beam energy spreads are limited by

$$\frac{\langle\gamma - \gamma_u\rangle}{\Delta\gamma} = \frac{\langle\gamma - \gamma_u\rangle}{\gamma_u\sqrt{2a_w a_r}}, \tag{52}$$

placing stringent energy spread limits in a FEL,

$$\frac{\langle\gamma - \gamma_u\rangle}{\gamma_u} \ll \sqrt{2a_w a_r} \simeq \frac{1}{2N_w}; \tag{53}$$

the far right-hand side is valid at optimum FEL efficiency when the wiggler length approximately equals half the bounce period of a trapped electron. Comparison between Eqs. (51) and (53) shows that DEL operation tolerates relaxed beam quality compared to its FEL counterpart over a wide parameter range. Also, in a DEL the required beam quality is not affected by the wiggler length as in a FEL.

Saturation in a DEL is caused by the off-axis walk of the emitting electrons. One can obtain an upper limit in electronic efficiency by counting the maximum number of radiation quanta that can be emitted by the same electron. Since a

GC shift $\delta X = \hbar\omega_r/eE_o$ accompanies each emission there are $N_\omega \simeq w_o/\delta X$ photons emitted before the electron moves outside the radiation envelope. The small signal electronic efficiency therefore is

$$\eta = \frac{N_\omega \hbar\omega_r}{mc^2(\gamma_u - 1)} = \frac{eE_o w_o}{mc^2(\gamma_u - 1)}. \tag{54}$$

In terms of the applied dc voltage $E_o = V_o/2D$ and defining the beam voltage as $eV_b = mc^2(\gamma_u - 1)$, the efficiency formula is recast as

$$\eta = \frac{w_o}{2D}\frac{V_0}{V_b}. \tag{55}$$

In a FEL, on the other hand, the electron velocity recoil causes a frequency shift $\delta\omega \simeq \omega_r(\hbar\omega_r/\gamma_u\gamma_o mc^2)$ from resonance per emission. The total number of photons emitted before the electron shifts outside the resonance width $\Delta\omega \simeq \omega_r/2\gamma_u^2 N_w$ is $N_\omega = \Delta\omega/\delta\omega$ and yields the following upper limit:

$$\eta = \frac{N_\omega \hbar\omega_r}{mc^2(\gamma_u - 1)} \simeq \frac{1}{2N_w}. \tag{56}$$

According to Eq. (55) the DEL efficiency, determined only by the ratios of the beam-to-cavity diameter and applied-to-beam voltage, can be quite high; in fact, it is limited only by the maximum voltage gradient that can be applied between the plates. The FEL efficiency, on the other hand, is inherently small since the number of wiggler periods N_w must be large to achieve a significant per-pass gain. The unfavorable inverse relation between per-pass gain and efficiency is absent in DELs.

High-power DEL operation is examined elsewhere [5] using the classic equations for the resonant electron response. It is shown there by computing the energy exchange rate $\mathbf{j}\cdot\mathbf{E}_r$ that the increase in the radiation energy exactly equals the change in the ES energy of the beam ''center of charge'' $\langle X\rangle$,

$$GP_r = \Gamma_e eE_o\langle\Delta X\rangle = I_b E_o\langle\Delta X\rangle, \tag{57}$$

where $\Gamma_e = I_b/e$ is the electron flux in the beam. Beam-radiation overlapping can be prolonged by the proper tilt of the resonator axis relative to the drift direction. The angle of tilt $\tan\theta = \langle\Delta X\rangle/L$ is related to the gain (54) by

$$\tan\theta = \frac{G(\Delta\omega, L)P_r}{I_b E_o L}. \tag{58}$$

Hence ''tapering'' in a DEL is controlled simply by the tilt angle between the drifting beam and the radiation beam. A remarkable DEL feature is that at high-power operation, when $a_r > a_w$, the GC excursion δX is unbounded. Unlike the pendulum-type FEL dynamics[3], there is no trapped particle island and no ''turning around the bucket'' for the X motion. Electrons keep moving along the dc electric field converting their potential energy into radiation [5] until they either shift outside the radiation envelope, hit the wiggler surface, or exit the interaction space.

Finally, we briefly touch upon another interesting possibility, namely, the DEL operation at a higher frequency resulting from the relativistic upshifting of the sum of the wiggler period plus the cyclotron frequency. By allowing transitions between different cyclotron states $\delta n = \mp 1$ the energy-momentum balance gives the new selection rules

$$\pm\hbar\omega_r = \mp\gamma_u\hbar\hat{\Omega} + eE_o\delta X, \tag{59a}$$

$$\pm\hbar(k_r + k_w) = \mp\gamma_u\frac{\hbar\hat{\Omega}}{c}\frac{u}{c} + m\Omega\,\delta X. \tag{59b}$$

An increment of electrostatic energy plus a cyclotron oscillation quantum are simultaneously converted into a photon during emission and vise versa. The drift kinetic energy and momentum are invariant, thus transition is again accompanied by a GC shift $\pm\Delta = \pm\hbar(\omega - \Omega)/eE_o$. The emitted radiation is centered around

$$\omega_r = \left(1 + \frac{u}{c}\right)\gamma_u^2\left(k_w u + \frac{\Omega}{\gamma_u^2}\right). \tag{60}$$

Everything else being the same, the ''drift-cyclotron'' frequency (60) is higher than the pure-drift DEL, Eq. (21b), by a factor $1 + \Omega/\gamma_u^2 k_w u$. The cyclotron emission adds two contributions to the gain. One is nonrelativistic, always stabilizing and symmetric in detuning $\Delta\omega = \omega_r - (k_r + k_w)u - \Omega/\gamma_u^2$. The relativistic contribution is antisymmetric in detuning and destabilizing when $\Omega/\gamma_u^2 > \omega_r - (k_r + k_w)u$. Detailed results of the gain computation will be reported in the near future.

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APPENDIX: FINITE THICKNESS BEAM

Earlier we considered the special case of a narrow beam, with a charge distribution of microscopic thickness d smaller than the transverse recoil distance $\Delta = \delta X$. To take up the general case of macroscopic thickness $d \gg \Delta$ consider a beam wave function made by a superposition of drifting eigenstates

$$\Psi(x) = \sum_q c_{X_q}\phi_n'(x - X_q)e^{iqz}e^{-i\mathcal{E}_{n,q}' t/\hbar}. \tag{A1}$$

A uniform charge density $e|\Psi(x)|^2 = en_o$ results within $X_o - d/2 \leqslant X_q \leqslant X_o + d/2$ if the guiding centers are uniformly distributed in X,

$$c_{X_q}^2 = n_o/d, \tag{A2}$$

provided the Larmor radius $\rho_n = \sqrt{(2n+1)\hbar/\hat{\Omega}}$ is also much smaller than $d/2$. Since cyclotron transitions are inhibited we assume the same cyclotron rotation number n for all electrons without loss of generality. Owing to the presence of the beam charge, the dc field and the potential relative to the beam center X_o are

$$E(X) = -E_o - 4\pi e n_o \overline{X},$$

$$eV(X) = eE_o\overline{X} + \tfrac{1}{2} m\omega_b^2 \overline{X^2} + eV_o \tag{A3}$$

where $\overline{X} \equiv X - X_o$, $\omega_b^2 = 4\pi e^2 n_o/m$ is the beam-plasma frequency and E_o, V_o are the values at X_o. The drifting eigenstates $\phi_n'(x - X_q)$ and eigenvalues and $\mathcal{E}_n'$ are perturbed by the beam self-potential $\frac{1}{2} m\omega_b^2 \overline{X}^2$. The cyclotron rotation frequency is modified into

$$\Omega^\dagger = \hat{\Omega}\sqrt{1 - \frac{\omega_b^2}{\Omega^2}} = \frac{\Omega}{\gamma_u}\sqrt{1 - \frac{\omega_b^2}{\Omega^2}} \tag{A4}$$

and the center X_q of the wave function is drifting at the modified local drift velocity

$$u(X_q) = c\,\frac{E(X_q)}{B} = u_0 + \frac{\omega_b^2}{\Omega}\overline{X_q}, \tag{A5}$$

where $u_o = cE_o/B$ is the drift velocity at the beam center. The energy and momentum are given by

$$\mathcal{E}'_{n,q} = \gamma_u \gamma_\perp' mc^2 - eE_o\overline{X_q} - \tfrac{1}{2}\omega_b^2 \overline{X_q^2} eV_o, \tag{A6}$$

$$P_z' = \hbar q = \gamma_u \gamma_\perp' m u(\overline{X_q}) - m\Omega\overline{X_q}, \tag{A7}$$

$$\gamma_\perp' = \sqrt{1 + 2(n + 1/2)\hbar\Omega^\dagger/mc^2 + (eA_w/mc^2)^2}. \tag{A8}$$

Notice that, according to Eq. (A5), $\gamma_u = 1/\sqrt{1 - u^2(\overline{X_q})/c^2}$ changes with the GC location as

$$\delta\gamma_u = \gamma_u^3 \beta_u \frac{\omega_b^2}{\Omega c}\,\delta\overline{X_q}. \tag{A9}$$

Equating the change in the energy (A6) and momentum (A7) during photon emission or absorption with $\mp\hbar\omega_r$ and $\mp\hbar(k_r + k_w)$, respectively, yields, using $\Delta \equiv \delta X_q$,

$$\pm\hbar\omega_r = eE(\overline{X_q})\left(-\gamma_\perp'\gamma_u^3\frac{\omega_b^2}{\Omega^2} + 1\right)\Delta + \tfrac{1}{2}\omega_b^2(2\overline{X_q}\Delta + \Delta^2), \tag{A10}$$

$$\pm\hbar(k_r + k_w) = m\Omega\left(-\gamma_\perp'\gamma_u^3\frac{\omega_b^2}{\Omega^2} + 1\right)\Delta, \tag{A11}$$

where we used Eq. (A9) for $\delta\gamma_u$. Relation (A10) yields a quadratic equation for the recoil

$$\Delta^2 + 2\mathcal{B}\Delta \pm \mathcal{C} = 0,$$

$$\mathcal{B} = \left\{ eE(\overline{X_q})\left(-\gamma_\perp'\gamma_u^3\frac{\omega_b^2}{\Omega^2} + 1\right) + m\omega_b^2\overline{X_q}\right\}\frac{1}{m\omega_b^2},$$

$$\mathcal{C} = \frac{2\hbar\omega_r}{m\omega_b^2}. \tag{A12}$$

Solving Eq. (A12) yields the recoil

$$\Delta_\pm = -\mathcal{B} + \sqrt{\mathcal{B}^2 \mp \mathcal{C}} \simeq \mp\frac{\mathcal{C}}{2\mathcal{B}} - \frac{1}{2\mathcal{B}}\left(\frac{\mathcal{C}}{2\mathcal{B}}\right)^2. \tag{A13}$$

In the low space charge limit, $\omega_b^2/\Omega^2 \ll 1/\gamma_u^3$, Eq. (A13) reduces to Eq. (35).

Taking the ratio between Eqs. (A10) and (A11) and neglecting the quadratic term Δ^2 eliminates Δ and yields the resonance condition

$$\omega_r - (k_r + k_w)u\,\frac{1 + (\omega_b^2/\Omega u)X + \gamma_\perp'\gamma_u^3\omega_b^2/\Omega^2}{1 - \gamma_\perp'\gamma_u^3\omega_b^2/\Omega^2} = 0. \tag{A14}$$

The peak frequency in the emission or absorption probability therefore varies with the GC location in the beam due to the shear in the GC velocity,

$$\omega_o(X) \simeq (k_r + k_w)u(X)\left(1 + \frac{(\omega_b^2/\Omega u)X}{1 - \gamma_\perp'\gamma_u^3\omega_b^2/\Omega^2}\right)$$
$$\simeq (k_r + k_w)u(X), \tag{A15}$$

where the far-right-hand side is valid for $\gamma_u^3 \gg 1$, $\Omega d/u$. The line shape factor for radiative transitions is now a function of both the frequency ω_r and the electron GC location X,

$$\Theta(\omega_r, \overline{X}) = \frac{\sin^2\xi}{\xi^2}, \tag{A16}$$

where

$$\xi(\omega_r, \overline{X}) = [\omega_r - \omega_o(\overline{X})]\frac{L}{2u} = \xi_o(\omega_r) - (k_r + k_w)\frac{\omega_b^2}{\Omega}\frac{L}{2u}\overline{X}, \tag{A17}$$

and $\xi_o = [\omega_r - (k_r + k_w)u_o]L/2u_o$ stands for the detuning at the beam center. The total emission probability is the sum of the individual emission probabilities over all the GC locations,

$$W = \int_{-d/2}^{d/2} d\overline{X} C_X^2 [W_+(\overline{X}) - W_-(\overline{X})]. \tag{A18}$$

Taking $W_\pm$ from Eq. (28) yields

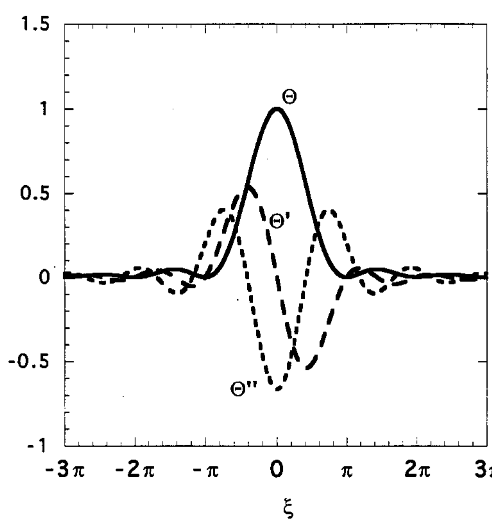

FIG. 6. The line shape function $\Theta(\xi)$ and its derivatives $\Theta'(\xi)$ and $\Theta''(\xi)$.

$$W_+ - W_- = \frac{1}{\hbar^2}\left\{\frac{\Delta_+ - \Delta_-}{2}(A^2)' + \frac{\Delta_+^2 - \Delta_-^2}{2}\right.$$
$$\times\left[(A')^2 - A^2\frac{(n+1/2)m\Omega^\dagger}{\hbar} - AA''(n+\tfrac{1}{2})^2\right]\Bigg\}$$
$$\left.\times\left(\frac{L}{u}\right)^2 \overline{\Theta(\omega_r, X)},\qquad \text{(A19)}$$

where, as usual, $A(\overline{X}) \equiv a_w a_r (mc^2/4\gamma_\perp \gamma_u)\cosh[k_w(X_o + \overline{X})]$ and $(') \equiv d/dX$.

For a beam of width a in the y direction, the electron flux is given by

$$a n_o \int_{-d/2}^{d/2} d\overline{X}\left(u_o - \frac{\omega_b^2}{\Omega}\overline{X}\right) = a n_o u_o, \qquad \text{(A20)}$$

thus one can apply the gain formula (36) with surface charge density $\sigma = e n_o d$ and Eqs. (A18) and (A19) for $W_+ - W_-$. The contributions from the terms $\Delta_\pm^2$ disappear in the classical limit $\hbar \to 0$. Assuming a beam thickness much smaller than the wiggler wavelength $k_w d \ll 1$ one can factor out all terms but the line shape factor outside the integral (A18), yielding

$$G = G_0 \frac{1}{d}\int_{-d/2}^{d/2} d\overline{X}\,\frac{\Theta(\xi(\overline{X}))}{\Theta(\xi_o)}. \qquad \text{(A21)}$$

Above, G_0 is the gain that would result for a zero-thickness beam of the same current placed at $X = X_o$. Notice that all the space-charge effects are buried in $\Theta(\xi(X))$. By expanding the argument $\xi(\omega_r, X)$ according to Eq. (A17), and letting $\Theta(\xi(X)) = \Theta(\xi_o) + \Theta'(\xi_o)(k_r + k_w)L\omega_b^2/(\Omega u_o)\overline{X} + \frac{1}{2}\Theta''(\xi_o)[(k_r + k_w)L\omega_b^2/(\Omega u_o)]^2 X^2$ and substituting in the rhs of Eq. (A21) yields

$$G = G_0\left\{1 + \frac{1}{6}\left(\frac{\omega_o}{\Omega}\frac{\omega_b^2 L d}{2u_o^2}\right)^2 \frac{\Theta''(\xi_o)}{\Theta(\xi_o)}\right\}, \qquad \text{(A22)}$$

$$G_0 = \frac{I_b}{2I_0}\frac{ck_w}{\beta_u \Omega}\frac{a_w^2}{\gamma_u^2\gamma_\perp^2}\frac{1}{(k_r w_o)^2}\left[\frac{1}{2}\left(\frac{1}{a_w^2}\frac{\partial a_w^2}{\partial(k_w X_o)}\right)\right]$$
$$\times\left(\frac{k_r L}{\beta_u}\right)^2 \Theta(\xi_o). \qquad \text{(A23)}$$

The function $\Theta''(\xi_o)$ plotted in Fig. 6 is negative at synchronism. Thus, in the wide beam case $d \gg \Delta$, ρ the space charge contribution to gain is negative (stabilizing) and of order $(\sigma/E_o)^2$, instead of the positive (destabilizing) contribution of order σ/E_o for the narrow beam limit $d \ll \Delta$, ρ. The gain remains symmetric in respect to the frequency detuning, measured relative to the central drift velocity $\Delta\omega = \omega_r - (k_r + k_w)u_o$, since $\Theta''(\xi_o)$ is even in ξ_o [the contribution from $\Theta'(\xi_o)$ that is odd in detuning vanished during the X integration over the beam thickness].

The magnitude of the space-charge coefficient in Eq. (A22) is better estimated by letting $u_o \simeq c = \omega_r/k_r$ where $\omega_r \simeq \omega_o = 2\gamma_{u_o}^2 k_w u_o$, yielding

$$\left(\frac{\omega_o}{\Omega}\frac{\omega_b^2 L d}{2u_o^2}\right)^2 = (2\pi)N_w k_r d\,\frac{k_w c}{\Omega}\frac{\omega_b^2}{(k_w c)^2}. \qquad \text{(A24)}$$

Taking the beam thickness equal to the radiation waist $k_r d = 2\pi w_o/\lambda_r$ and for wiggler period of 1 cm and $k_w c/\Omega \sim 1$ the coefficient (A24) is near unity when $\omega_b^2 \sim 4c^2/(N_w w_o/\lambda_r)$. For typical $N_w \sim 10^2$, $w_o/\lambda_r = 10$ follows $n_o \sim 6\times 10^8$ cm^{-3} which corresponds to current density $J_b \simeq e n_o c \sim 2.88$ A/cm^2.

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